BORDER COLLISION OF NON-HYPERBOLIC FIXED POINTS

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Abstract: We report novel results on three codimension-two bifurcations in nonsmooth systems, namely the border-fold, border-flip, and border-Neimark-Sacker. These results, obtained in the discrete-time framework, are easily applicable in the analysis of Poincaré maps of periodic solutions in continuous-time. Three models, taken from different fields of science and engineering, are then briefly described and used as test benches for the theoretical results.

Keywords: bifurcation, border collision, codimension-two, non-hyperbolic, nonsmooth

1. INTRODUCTION

Knowledge of the geometry of bifurcation curves around codimension-two points is a key ingredient to efficiently derive complex bifurcation diagrams. In the domain of smooth dynamical systems, the unfolding of the most common codimension-two points is well known (see, e.g., [Kuznetsov, 2004]), and this knowledge is exploited in continuation software for the automatic switching of bifurcation branches at these points (see, e.g., [Dhooge et al., 2002, Meijer et al., 2009]). With the advent of new continuation packages for nonsmooth systems [Dercole and Kuznetsov, 2005, Thota and Dankowicz, 2008], new effort is required to extend these results to the discontinuous case (see [di Bernardo et al., 2008, Kowalczyk and di Bernardo, 2005, Kowalczyk et al., 2006] for an overview of the subject). In this paper, we collect some results on border collision bifurcations of non-hyperbolic fixed points. This is a codimension-two bifurcation that takes place when a non-hyperbolic fixed point of a discrete-time nonsmooth system hits a discontinuity boundary separating smooth regions of the state space. Rather than aiming at a complete unfolding of the these bifurcations, which would require specific assumptions on the system's equations on both sides of the involved boundary [di Bernardo et al., 2008], we concentrate on the geometric features that are common to all scenarios. These results apply as well to codimension-two bifurcations of non-hyperbolic periodic orbits in continuous-time systems (hybrid, impacting, piecewise smooth, etc.), through the analysis of the corresponding Poincaré maps.

In what follows, we report the theoretical results and a sketch of the procedure to derive them, then we analyse three applications. Complete proofs are going to be detailed in a full length paper.

2. CODIMENSION-TWO NORMAL FORMS

Smooth systems theory tells us that a fixed point can be non-hyperbolic in three generic ways, which are distinguished based on the eigenvalues of the Jacobian of the system's map evaluated at the fixed point. Generically, if one eigenvalue is equal to 1 we have a fold bifurcation, if it is equal to -1 we have a flip bifurcation, if two complex conjugate eigenvalues are on the unit circle we have a Neimark-Sacker bifurcation. In the case of a nonsmooth system having a fixed point in the interior of a smooth region S, this scenario is unchanged. Consequently, when a non-hyperbolic fixed point of the nonsmooth system hits the boundary of S, we can distinguish three cases, a border-fold, a border-flip, and a border-Neimark-Sacker bifurcation.

To understand the local geometry of bifurcation curves around these codimension-two points, we proceed through three steps. First, we reduce the map of the nonsmooth system, restricted to region S, to normal form. This is done in a similar way as for smooth systems by considering the map restricted to a parameter-dependent centre manifold, which is one-dimensional in the fold and flip cases, and two-dimensional in the Neimark-Sacker case, and reducing the terms in the Taylor expansion of the map with appropriate changes of variable and parameter (i.e, see [Kuznetsov, 2004]). While the system's map in S has the form

$$z \mapsto F(z,\alpha), \quad z \in \mathbf{R}^n, \alpha \in \mathbf{R}^2,$$

where z is the state vector and α is the parameter vector, its restriction to the centre manifold has the form

$$u \mapsto f(u,\alpha), \tag{1}$$

where u is a one- or two-dimensional vector in the centre manifold. The normal form, say

$$v \mapsto f^{NF}(v,\beta), \tag{2}$$

is then obtained through changes of variable and parameters. Since all parameter changes are quasi-identities, $\beta = 0$ when $\alpha = 0$. We are free to assume that the codimension-two bifurcation occurs when $\alpha = 0$, and that when $\alpha = 0$ the fixed point is located at $z = 0$. Then generically near $\alpha = 0$ the centre manifold transversally intersects the discontinuity boundary in a neighbourhood of $z = 0$. This allows us to write the restriction of the discontinuity boundary to the centre manifold as the zero-set of a function $h(u,\alpha)$ with nonsingular Jacobian at $(0,0)$, that is,

$$\Sigma = \{u : h(u,\alpha) = 0\} \tag{3}$$

with $h_u(0,0) \neq 0$.

As a second step, we find the expression of the discontinuity boundary (3) in the new variables and parameters, i.e.,

$$\Sigma = \{v : h^{\mathrm{NF}}(v,\beta) = 0\}. \tag{4}$$

Finally, third step, we analyse the interaction of the normal form map (2) with the discontinuity boundary (4), and we find local asymptotics for the bifurcation curves emanating from $\beta = 0$ in terms of (β_1,β_2)-expansions.

On the whole, we obtain the following results for the three cases, where the superscript 0 stands for evaluation at $(u,\alpha) = (0,0)$:

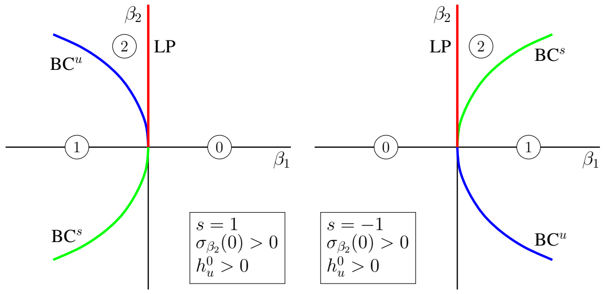

Fig. 1. Border-fold bifurcation. Bifurcation curves: LP, fold (limit point, red); BC^s, border collision of $\bar{v}^s$ (green), BC^u, border collision of $\bar{v}^u$ (blue). Region labels: 0, no fixed point in $V^-(\beta) = \{v : h^{\mathrm{NF}}(v,\beta) < 0\}$; 1, $\bar{v}^s$ (left) or $\bar{v}^u$ (right) is the only fixed point in $V^-(\beta)$; 2, both fixed points $\bar{v}^{s,u}$ lie in $V^-(\beta)$.

2.1 *Fold case*

Assume that $f(0,0) = 0$ (fixed point condition), that $f_u(0,0) = 1$ (fold condition), and that the following genericity conditions are met

(C.1) $h_u^0 \neq 0$,
(C.2) $f_{uu}^0 \neq 0$,
(C.3) $h_{\alpha_1}^0 f_{\alpha_2}^0 + \dfrac{h_u^0 f_{\alpha_1}^0 f_{u\alpha_2}^0}{f_{uu}^0} \neq h_{\alpha_2}^0 f_{\alpha_1}^0 + \dfrac{h_u^0 f_{\alpha_2}^0 f_{u\alpha_1}^0}{f_{uu}^0}$.

Then, locally to $\beta = 0$, there are a fold curve of equation

$$\beta_1 = 0, \beta_2 \gtrless 0 \tag{5}$$

and a border collision curve of equation

$$\pm\sqrt{-s\beta_1} \simeq \sigma_{\beta_2}(0)\beta_2, \tag{6}$$

where $s = \pm 1$ is a coefficient of the normal form [Kuznetsov, 2004],

$$\sigma_{\beta_2}(0) = \frac{1}{h_u^0 \|f_\alpha^0\|^2}\left(\left(h_{\alpha_1}^0 - \frac{h_u^0 f_{u\alpha_1}^0}{f_{uu}^0}\right) f_{\alpha_2}^0 - \left(h_{\alpha_2}^0 - \frac{h_u^0 f_{u\alpha_2}^0}{f_{uu}^0}\right) f_{\alpha_1}^0\right),$$

and the inequality sign in (5) depends on the signs of $\sigma_{\beta_2}(0)$ and h_u^0. Equation (6) describes a parabola whose two branches correspond to the border collisions of the two fixed points $\bar{v}^s$ (stable) and $\bar{v}^u$ (unstable) of map (1), that appear at the fold bifurcation. The signs of s, $\sigma_{\beta_2}(0)$, and h_u^0 depend on functions f and h and give place to eight possible cases, only two of which (shown in Fig. 1) have qualitatively different bifurcation diagrams.

2.2 *Flip case*

Assume that $f(0,\alpha) = 0$ for small $\|\alpha\|$ (fixed point condition), that $f_u(0,0) = -1$ (flip condition),

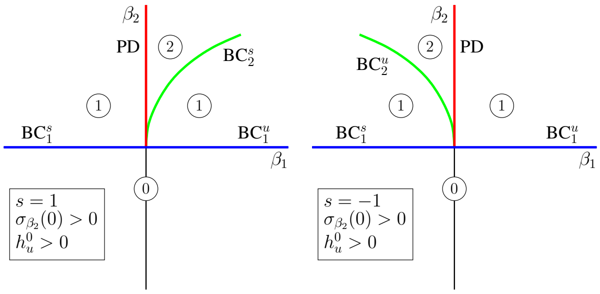

Fig. 2. Border-flip bifurcation. Bifurcation curves: PD, flip (period doubling, red); $\mathrm{BC}_1^{s,u}$, border collision of the fixed point $v = 0$ (stable and unstable branches, blue); $\mathrm{BC}_2^{s,u}$, border collision of the stable or unstable period-two fixed point. Region labels: 0, no fixed point in $V^-(\beta) = \{v : h^{\mathrm{NF}}(v,\beta) < 0\}$; 1, $v = 0$ is the only fixed point in $V^-(\beta)$; 2, the fixed point $v = 0$ coexists in $V^-(\beta)$ with the period-two fixed point.

and that the following genericity conditions are met

(C.1) $h_u^0 \neq 0$,
(C.2) $\frac{1}{2}(f_{uu}^0)^2 + \frac{1}{3}f_{uuu}^0 \neq 0$,
(C.3) $h_{\alpha_1}^0 f_{u\alpha_2}^0 \neq h_{\alpha_2}^0 f_{u\alpha_1}^0$.

Then, locally to $\beta = 0$, there are a flip curve of equation

$$\beta_1 = 0, \beta_2 \gtrless 0, \tag{7}$$

a border collision curve of the period-one fixed point of equation

$$\beta_2 = 0,$$

and a border collision of the period-two fixed points of equation

$$\pm\sqrt{s\beta_1} \simeq \sigma_{\beta_2}(0)\beta_2, \tag{8}$$

where $s = \pm 1$,

$$\sigma_{\beta_2}(0) = -\frac{\sqrt{|c(0)|}}{h_u^0}$$

with $c(0) = (1/4)(f_{uu}^0)^2 + (1/6)f_{uuu}^0$, and the inequality sing in (7) depends on the signs of $\sigma_{\beta_2}(0)$ and h_u^0. The signs of s, $\sigma_{\beta_2}(0)$, and h_u^0 depend on functions f and h. Overall we have two qualitatively different portraits, depicted in Fig. 2.

2.3 *Neimark-Sacker case*

Assume that $f(0,\alpha) = 0$ for small $\|\alpha\|$ (fixed point condition), and that the 2×2 Jacobian $f_u(0,0)$ has eigenvalues $\lambda(0)$ and $\bar{\lambda}(0)$ (the overbar stands for complex conjugation), with

$$\lambda(\alpha) = (1 + g(\alpha))e^{i\theta(\alpha)},$$

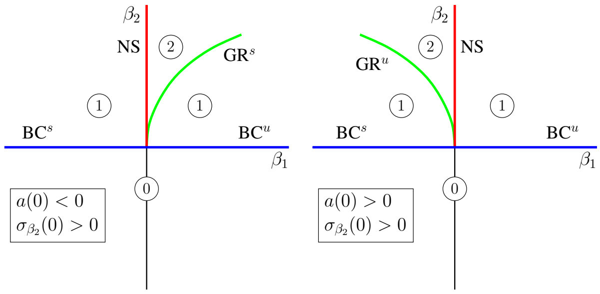

Fig. 3. Border-NS bifurcation. Bifurcation curves: NS, Neimark-Sacker (red); $\mathrm{BC}^{s,u}$, border collision of the fixed point $v = 0$ (stable and unstable branches, blue); $\mathrm{GR}^{s,u}$, grazing of the stable or unstable torus (green). Region labels: 0, no fixed point or invariant curve in $V^-(\beta) = \{v : h^{\mathrm{NF}}(v,\beta) < 0\}$; 1, $v = 0$ is a fixed point in $V^-(\beta)$ and there is no invariant curve, or part of it does not lie in $V^-(\beta)$; 2, both the fixed point $v = 0$ and the invariant curve lie in $V^-(\beta)$.

where $g(0) = 0$ and $\theta(0) = \theta_0$ (Neimark-Sacker, NS, condition). Moreover, assume the following genericity conditions

(C.1) $h_u^0 \neq 0$,
(C.2) $e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$,
(C.3) the NS is super- or subcritical at $\alpha = 0$,
(C.4) $h_{\alpha_1}^0 g_{\alpha_2}(0) \neq h_{\alpha_2}^0 g_{\alpha_1}(0)$.

Then, there are a Neimark-Sacker curve of equation

$$\beta_1 = 0, \beta_2 \gtrless 0 \tag{9}$$

and a border collision curve of the fixed point of equation

$$\beta_2 = 0,$$

while the invariant torus grazes the discontinuity boundary along a curve of equation

$$\sqrt{-\beta_1/a(0)} \simeq \sigma_{\beta_2}(0)\beta_2. \tag{10}$$

Here, $a(0)$ is a coefficient of the NS normal form and

$$\sigma_{\beta_2}(0) = -\frac{1}{2h_u^0 \mathrm{Re}(q(0)e^{i\varphi_h})},$$

where

$$\varphi_h = \arctan_{2\pi}\left(h_u^0\mathrm{Re}(q(0)), -h_u^0\mathrm{Im}(q(0))\right),$$

and vector $q(0)$ is the right eigenvector of the Jacobian of (1) at $\alpha = u = 0$, associated with $\lambda(0)$ (normalised so that the scalar product with the left eigenvector associated with $\bar{\lambda}(0)$ is 1). Once again, the inequality sign in (9) depends on the sings of $a(0)$ and $\sigma_{\beta_2}(0)$ which, in turn, depend on functions f and h, and give place to the two qualitatively different portraits depicted in Fig. 3.

3. EXAMPLES

We now present three examples, one for each of the three codimension-two bifurcations analysed in the previous section. The three examples deal with different classes of nonsmooth systems (an impacting system, a hybrid system, and a piecewise smooth system) and describe interesting applications in different fields of science and engineering (ecology, social sciences, and mechanics). The three systems we analyse are continuous time but, as we wrote before, the analysis in a neighbourhood of a periodic solution can be cast into the discrete-time framework by recurring to Poincaré maps.

An impacting model of forest fires

For an example of border-fold bifurcation, we consider the forest fire impacting model presented in [Dercole and Maggi, 2005, Maggi and Rinaldi, 2006]. The model describes the vegetational growth with the following two (smooth) ODEs:

$$\dot{B} = r_B B \left(1 - \frac{B}{K_B}\right) - \alpha BT,$$
$$\dot{T} = r_T T \left(1 - \frac{T}{K_T}\right),$$

one for the surface layer (bush, B) and one for the upper layer (trees, T). Fire episodes are represented by instantaneous events (impacts), that occur when the biomasses (B, T) of the two layers reach one of three specified impacting boundaries: a bush ignition threshold $\rho_B K_B$ triggering bush-only fires that map the bush biomass to $\lambda_B \rho_B K_B$, $0 < \lambda_B, \rho_B < 1$; a tree ignition threshold $\rho_T K_T$ triggering trees-only fires that map the trees biomass to $\lambda_T \rho_T K_T$, $0 < \lambda_T, \rho_T < 1$; and the segment connecting points $(\sigma_B K_B, \rho_T K_T)$ and $(\rho_B K_B, \sigma_T K_T)$, $0 < \sigma_B < \rho_B$, $0 < \sigma_T < \rho_T$, triggering mixed fires with post-fire conditions suitably assigned as a function of pre-fire conditions (see [Maggi and Rinaldi, 2006] for more details).

For the parameter setting $r_1 = 0.375$, $r_2 = 0.0625$, $\alpha = 0.43$, $K_B = K_T = 1$, $\rho_B = 0.85$, $\rho_T = 0.93$, $\lambda_B = 0.03$, $\lambda_T = 0.01$, $\sigma_B = 0.61$, $\sigma_T = 0.3$ (corresponding to Mediterranean forests), the system is characterised by a globally stable period-one cycle composed of a growth orbit and a mixed fire. Numerical continuation (by means of AUTO07P [Doedel et al., 2007]) of the cycle in the parameter plane (ρ_B, ρ_T) identifies two (codimension-one) bifurcations: a fold (red curve in Fig. 4) and a grazing of the growth orbit with the bush ignition threshold (blue curve). The two curves merge together at the border-fold

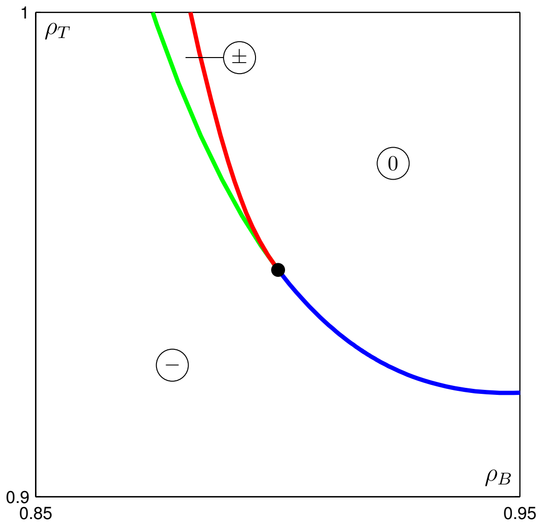

Fig. 4. Example of border-fold bifurcation. Bifurcation curves: fold (red); border collision of the period-one stable cycle (blue); border collision of the period-one unstable cycle (green). Region labels as in Fig. 1.

bifurcation (black) point and, as predicted, the grazing bifurcation of the unstable cycle involved in the fold (green curve) emanates tangentially from the codimension-two bifurcation point.

A hybrid model of two-party democracies

For an example of border-flip bifurcation, we consider the hybrid model presented in [Colombo and Rinaldi, 2008] for describing the dynamics of two-party democracies. The model describes the evolution of the size of two lobbies (of sizes L_D and L_R), one associated to each party (parties D and R, respectively), and assumes that the individuals belonging to the lobby of the party at the government erode the welfare (W) at a rate proportional to the size of the lobby; a lobby can grow only as long as its party is at the government, and decays otherwise; a small fraction of the lobbyists not at the government defect and switch to the other lobby; elections are held once every T years, and people vote for the party that has the least damaging lobby at the time of the elections. Altogether, the dynamics is captured by two sets of ODEs, namely

$$\dot{W} = r(1 - W - a_D L_D)W,$$
$$\dot{L}_D = (e_D a_D W - d_D)L_D + k_R L_R,$$
$$\dot{L}_R = (-d_R - k_R)L_R,$$

when the D-party is at the government, and

$$\dot{W} = r(1 - W - a_R L_R)W,$$
$$\dot{L}_D = (-d_D - k_D)L_D,$$
$$\dot{L}_R = (e_R a_R W - d_R)L_R + k_D L_D,$$

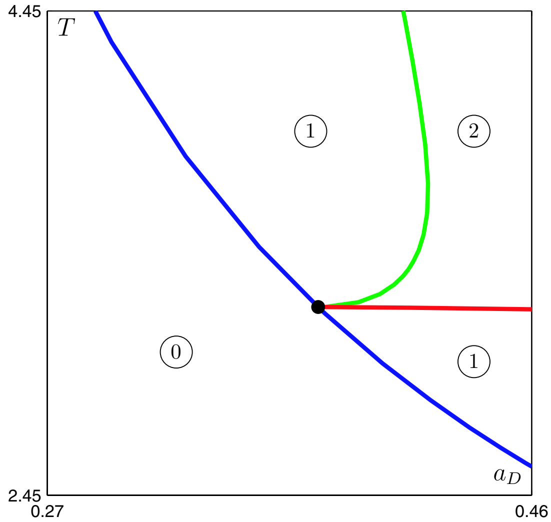

Fig. 5. Example of border-flip bifurcation. Bifurcation curves: flip (red); border collision of the period-one cycle (blue); border collision of the period-two cycle (green). Region labels as in Fig. 2.

when the R-party is at the government. Here, r is the intrinsic growth rate of the welfare, a represents the aggressiveness of a lobby, e is the recruitment coefficient of a lobby, and d and k are respectively the rate at which individuals abandon the lobbies or defect. In the region of the state space where $a_D L_D < a_R L_R$ ($a_D L_D > a_R L_R$) the D-lobby (R-lobby) is less damaging and thus wins the elections. The condition $a_D L_D = a_R L_R$ therefore defines the discontinuity boundary (see [Colombo and Rinaldi, 2008] for more details).

In the (a_D, T) plane, with parameters $a_R = 1$, $r = 0.2$, $e_D = e_R = 6$, $d_D = d_R = 1.8$, $k_D = k_R = 0.06$, the system has a very complex bifurcation diagram (see for example Fig. 1 in [Colombo and Rinaldi, 2008]). In particular, near $a_D = 0.38, T = 3.2$, a flip (red curve in Fig. 5) and a border collision (blue curve) of a simple period-$2T$ cycle meet at the border-flip (black) point and, as predicted by the analysis carried out in Sect. 4, a border collision of the period-4T cycle (green curve) emanates from the codimension-two point tangentially to the flip curve.

A piecewise smooth model of railway wheelset dynamics

For an example of border-NS bifurcation, we consider a two degrees of freedom piecewise smooth model of a suspended railway wheelset with dry friction dampers, subject to a sinusoidal disturbance representing the deformations of the track. The model is based on that presented in [Knudsen et al., 1992, True and Asmund, 2003], where the track deformation was not taken into account, and its analysis will be published elsewhere. Since a detailed explanation of the equations and parameters goes well beyond the scope of this paper, here we only report the equations and describe a few key parameters (see [Knudsen et al., 1992] and [True and Asmund, 2003] for the details). The model consists of the following piecewise smooth equations:

$$
\begin{aligned}
\dot{x}_1 &= \tilde{x}_2, \\
\dot{x}_2 &= \frac{1}{m}(-2F_x - 2K_s\tilde{x}_1 - \mathrm{sign}(x_2)\mu), \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= \frac{1}{I}(-2AF_y),
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{x}_1 = x_1 + a\sin(\omega t), \quad \tilde{x}_2 = x_2 + a\omega\cos(\omega t), \\
&\mu = (\mu_d(1-\mathrm{sech}(\alpha\tilde{x}_2)) + \mu_s \mathrm{sech}(\alpha\tilde{x}_2)), \\
&F_x = \frac{\xi_x F_r}{\Psi\xi_r}, \quad F_y = \frac{\xi_y F_r}{\Phi\xi_r}, \\
&F_r = \begin{cases} \xi_r C\left(1 - \dfrac{C\xi_r}{3\mu_t} + \dfrac{C^2\xi_r^2}{27\mu_t^2}\right) & \text{if } C\xi_r < 3\mu_t, \\ \mu_t \text{ otherwise,} \end{cases} \\
&\xi_x = \frac{\tilde{x}_2}{V} - x_3, \quad \xi_y = \frac{Ax_4}{V} + \frac{\lambda\tilde{x}_1}{r_0}, \\
&\xi_r = \sqrt{\left(\frac{\xi_x}{\Psi}\right)^2 + \left(\frac{\xi_y}{\Phi}\right)^2}.
\end{aligned}
$$

Here $\omega = 2\pi V/l$, a and l are the amplitude and wavelength of the sinusoidal disturbance, V is the speed of the wheelset, and λ measures the conicity of the wheels. The system's state space is therefore partitioned in four regions, depending on the signs of x_2 and of $C\xi_r - 3\mu_t$, so that $x_2 = 0$ and $C\xi_r = 3\mu_t$ define two discontinuity boundaries.

The system's dynamics was studied, with TC-HAT [Thota and Dankowicz, 2008], in the (V, λ) plane, with the following values of the parameters: $m = 1022$, $K_s = 1e6$, $I = 678$, $A = 0.75$, $a = 0.001$, $\mu_d = 1000$, $\alpha = 50$, $\mu_s = 1200$, $\Psi = 0.54219$, $\Phi = 0.60252$, $C = 6.5630e6$, $\mu_t = 10000$, $r_0 = 0.4572$, $l = 10$. For large values of V, a grazing with the boundary $x_2 = 0$ and a NS take place (blue and red in Fig. 6), and meet at the border-NS (black) point. Then, by systematically evaluating 1000 iterations (after transient) of the Poincaré map of the torus on a suitable cross-section, and by continuing the line on which the obtained torus image grazes the discontinuity boundary induced on the cross-section by the system's dynamics, we were able to trace an approximation of the grazing curve of the torus (green in Fig. 6). As predicted, the curve emanates from the codimension-two point tangentially to the NS curve.

4. CONCLUSIONS

We have presented some general results on the geometry of bifurcation curves around three codimension-

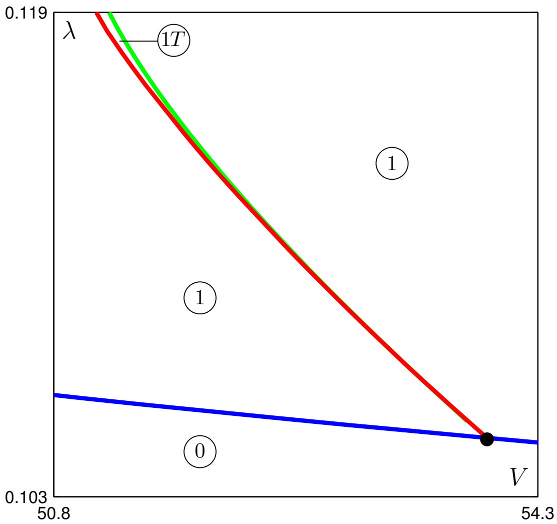

Fig. 6. Example of border-NS bifurcation. Bifurcation curves: Neimark-Sacker (red); border collision of the period-one cycle (blue); border collision of the torus (green). Region labels as in Fig. 3.

two bifurcations in nonsmooth systems, namely the border-fold, the border-flip, and the border-Neimark-Sacker. Rather than aiming at the complete unfolding of the dynamics of a particular class of nonsmooth systems, we have focused on those results which are general to all scenarios. Our approach applies to continuous-time as well as discrete-time systems, and basically consists of the analysis of a discrete-time (Poincaré) map defined only on one side of a boundary in its state space.

Of course, the weakness of this approach is that it cannot provide the complete unfolding of the bifurcation, but its power resides in its generality: as shown in the three examples that we have reported, it applies to a very broad class of nonsmooth systems and it may be relevant in various fields of science and engineering.

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